Generic programming with fixed points for mutually recursive datatypes

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Datatype-generic programming

- Write functions that depend on the structure of datatypes.
- Equality, parsing, . . .
- Traversing data structures, collecting or modifying items.
- Type-indexed data types: tries, zippers.
This talk

▶ Yet another (datatype-)generic programming library for Haskell.
▶ Gives you access to recursive positions, i.e., it is easy to write a generic fold/catamorphism.
▶ Allows you to define type-indexed datatypes, e.g., zippers.
▶ Applicable to a large class of datatypes, in particular mutually recursive datatypes.
This talk

▶ Yet another (datatype-)generic programming library for Haskell.
▶ Gives you access to recursive positions, i.e., it is easy to write a generic fold/catamorphism.
▶ Allows you to define type-indexed datatypes, e.g., zippers.
▶ Applicable to a large class of datatypes, in particular mutually recursive datatypes.
What is in a generic programming library?

- Represent datatypes generically.
- Map between user types and their representations.
- Define functions based on representations.
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- Represent datatypes generically.
- Map between user types and their representations.
- Define functions based on representations.

We focus on the first: **generic view or universe**.
PolyP (Jansson and Jeuring 1997)

The first approach to generic programming in Haskell:

- Datatypes are represented as fixed points of sums of products.
Example

```
data Expr = Const Val
    | If Expr Expr Expr
```
Example

\[
\begin{align*}
\text{data} \ \text{Expr} & \quad = \quad \text{Const} \ \text{Val} \\
& \quad | \quad \text{If} \quad \text{Expr} \ \text{Expr} \ \text{Expr}
\end{align*}
\]

As a functor:

\[
\begin{align*}
\text{data} \ \text{ExprF} \ e & \quad = \quad \text{ConstF} \ \text{Val} \\
& \quad | \quad \text{IfF} \quad e \quad e \quad e
\end{align*}
\]

\[\text{type} \ \text{Expr}' \quad = \quad \text{Fix} \ \text{ExprF}\]

\[\text{data} \ \text{Fix} \ f \quad = \quad \text{In} \ (f \ (\text{Fix} \ f))\]
Example

```haskell
data Expr  = Const Val
            | If       Expr Expr Expr

As a functor:

type ExprF e  = Val
            | e e e

type Expr'    = Fix ExprF

data Fix f    = In (f (Fix f))
```
Example

```haskell
data Expr = Const Val
          | If Expr Expr Expr

As a functor:

type ExprF e = Val
              + e e e e

type Expr' = Fix ExprF

data Fix f = In (f (Fix f))
```
Example

\begin{align*}
\textbf{data} \ \text{Expr} & = \ \text{Const} \ \text{Val} \\
& \mid \ \text{If} \ \text{Expr} \ \text{Expr} \ \text{Expr} \\
\text{As a functor:} \\
\textbf{type} \ \text{Expr} & = \ \text{Val} \\
& + \ e \times e \times e \\
\textbf{type} \ \text{Expr}' & = \ \text{Fix} \ \text{Expr}F \\
\textbf{data} \ \text{Fix} \ f & = \ \text{In} \ (f \ (\text{Fix} \ f))
\end{align*}
Example

```haskell
data Expr = Const Val
           | If Expr Expr Expr

As a functor:

type ExprF = K Val
            :+: [I :×: I :×: I]

type Expr' = Fix ExprF

data Fix f = In (f (Fix f))
```
Combinators

```hs
data I r = I r
data K a r = K a
data U r = U -- for constructors with no arguments
data (f :+: g) r = L (f r) | R (g r)
data (f :*: g) r = f r :*: g r
```

Functors are of kind $* \rightarrow *$.

```hs
data Fix (f :: * -> *) = In (f (Fix f))
```
Writing a generic function

```haskell
class Functor f where
    fmap :: (a -> b) -> f a -> f b

instance Functor (K a) where
    fmap f (K x) = K x

instance Functor I where
    fmap f (I x) = I (f x)

-- instances for the other functor combinators
```
Writing a generic function

```haskell
class Functor f where
  fmap :: (a → b) → f a → f b

instance Functor (K a) where
  fmap f (K x) = K x

instance Functor I where
  fmap f (I x) = I (f x)

-- instances for the other functor combinators

fold :: Functor f ⇒ (f r → r) → Fix f → r
fold alg (In f) = alg (fmap (fold alg) f)
```
Summary of workflow

▶ Use a limited set of combinators to build functors (library).
▶ Express datatypes as fixed points of functors (user or Template Haskell).
▶ Express the equivalence using a pair of conversion functions (user or Template Haskell).
▶ Define functions (and datatypes) on the structure of functors (library).
▶ Enjoy generic functions on all the represented datatypes (user).
Limitation of the PolyP approach

Only regular datatypes can be represented.

```haskell
data Expr = Const Val
           | If Expr Expr Expr
```

Typical ASTs are not regular, but a family of several mutually recursive datatypes.
Limitation of the PolyP approach

Only regular datatypes can be represented.

\[
\textbf{data} \; \text{Expr} = \text{Const Val} \\
\quad \mid \text{If} \; \text{Expr} \; \text{Expr} \; \text{Expr} \\
\quad \mid \text{Bin} \; \text{Expr} \; \text{Op} \; \text{Expr}
\]
Limitation of the PolyP approach

Only regular datatypes can be represented.

\[
\text{data} \ \text{Expr} = \text{Const Val} \\
\quad | \quad \text{If} \ \text{Expr} \ \text{Expr} \ \text{Expr} \\
\quad | \quad \text{Bin} \ \text{Expr} \ \text{Op} \ \text{Expr} \\
\]

\[
\text{data} \ \text{Op} = \text{Add} | \text{Mul} | \text{Infix} \ \text{Expr} | \text{Flip} \ \text{Op}
\]
Limitation of the PolyP approach

Only regular datatypes can be represented.

\[
data \text{ Expr } = \text{ Const Val} \mid \text{ If } \text{ Expr Expr Expr} \mid \text{ Bin } \text{ Expr Op Expr}
\]

\[
data \text{ Op } = \text{ Add } \mid \text{ Mul } \mid \text{ Infix Expr } \mid \text{ Flip Op}
\]

Typical ASTs are not regular, but a family of several mutually recursive datatypes.
Classic attempts

\[
\text{data Expr} \quad = \quad \text{Const Val} \\
\quad | \quad \text{If} \quad \text{Expr} \quad \text{Expr} \quad \text{Expr}
\]

\[
\text{data ExprF e} \quad = \quad \text{ConstF Val} \\
\quad | \quad \text{IfF} \quad e \quad e \quad e
\]

\[
\text{type Expr'} \quad = \quad \text{Fix ExprF}
\]
Classic attempts

data Expr = Const Val
| If Expr Expr Expr
| Bin Expr Op Expr

data Op = Add | Mul | Infix Expr | Flip Op

data ExprF e o = ConstF Val
| IfF e e e
| BinF e o e

data OpF e o = AddF | MulF | InfixF e | FlipF o

type Expr' = Fix\_{2,0} ExprF OpF

type Op' = Fix\_{2,1} ExprF OpF
Kinds

\[ \text{Fix} :: (\ast \to \ast) \to \ast \]
Kinds

\[
\text{Fix} :: (\ast \to \ast) \to \ast
\]

\[
\text{Fix}_2,0 :: (\ast \to \ast \to \ast) \to (\ast \to \ast \to \ast) \to \ast
\]

\[
\text{Fix}_2,1 :: (\ast \to \ast \to \ast) \to (\ast \to \ast \to \ast) \to \ast
\]

\[
\text{Fix}_3,0 :: (\ast \to \ast \to \ast \to \ast) \to (\ast \to \ast \to \ast \to \ast) \to (\ast \to \ast \to \ast \to \ast) \to \ast
\]

\[
\text{Fix}_3,1 :: (\ast \to \ast \to \ast \to \ast) \to (\ast \to \ast \to \ast \to \ast) \to (\ast \to \ast \to \ast \to \ast) \to \ast
\]

\[
\text{Fix}_3,2 :: (\ast \to \ast \to \ast \to \ast) \to (\ast \to \ast \to \ast \to \ast) \to (\ast \to \ast \to \ast \to \ast) \to \ast
\]

\ldots
Kinds

\[ \text{Fix} \quad :: \quad (\star \to \star) \to \star \]

\[ \text{Fix}_{2,0} \quad :: \quad (\star \to \star \to \star) \to (\star \to \star \to \star) \to \star \]
\[ \text{Fix}_{2,1} \quad :: \quad (\star \to \star \to \star) \to (\star \to \star \to \star) \to \star \]

\[ \text{Fix}_{3,0} \quad :: \quad (\star \to \star \to \star \to \star) \to (\star \to \star \to \star \to \star) \to (\star \to \star \to \star \to \star) \to \star \]
\[ \text{Fix}_{3,1} \quad :: \quad (\star \to \star \to \star \to \star) \to (\star \to \star \to \star \to \star) \to (\star \to \star \to \star \to \star) \to \star \]
\[ \text{Fix}_{3,2} \quad :: \quad (\star \to \star \to \star \to \star) \to (\star \to \star \to \star \to \star) \to (\star \to \star \to \star \to \star) \to \star \]
Kinds

$\text{Fix} :: (\ast \to \ast) \to \ast$

$\text{Fix}_{2,0} :: (\ast \to \ast \to \ast) \to (\ast \to \ast \to \ast) \to \ast$

$\text{Fix}_{2,1} :: (\ast \to \ast \to \ast) \to (\ast \to \ast \to \ast) \to \ast$

$\text{Fix}_{3,0} :: (\ast \to \ast \to \ast \to \ast) \to (\ast \to \ast \to \ast \to \ast) \to (\ast \to \ast \to \ast \to \ast) \to \ast$

$\text{Fix}_{3,1} :: (\ast \to \ast \to \ast \to \ast) \to (\ast \to \ast \to \ast \to \ast) \to (\ast \to \ast \to \ast \to \ast) \to \ast$

$\text{Fix}_{3,2} :: (\ast \to \ast \to \ast \to \ast) \to (\ast \to \ast \to \ast \to \ast) \to (\ast \to \ast \to \ast \to \ast) \to \ast$

\ldots
Kinds (contd.)

\[
\begin{align*}
\text{Fix}_{2,0} &:: \ (\ast \to \ast \to \ast) \to (\ast \to \ast \to \ast) \to \ast \\
\text{Fix}_{2,1} &:: \ (\ast \to \ast \to \ast) \to (\ast \to \ast \to \ast) \to \ast
\end{align*}
\]
Kinds (contd.)

\[ \text{Fix}_{2,0} :: \left( * \rightarrow * \rightarrow * \right) \rightarrow \left( * \rightarrow * \rightarrow * \right) \rightarrow * \]
\[ \text{Fix}_{2,1} :: \left( * \rightarrow * \rightarrow * \right) \rightarrow \left( * \rightarrow * \rightarrow * \right) \rightarrow * \]

If we had tuples on the kind level:

\[ \text{Fix}_2 :: \left( *^2 \rightarrow * \right)^2 \rightarrow *^2 \]
Kinds (contd.)

\[ \text{Fix}_{2,0} :: (\ast \rightarrow \ast \rightarrow \ast) \rightarrow (\ast \rightarrow \ast \rightarrow \ast) \rightarrow \ast \]
\[ \text{Fix}_{2,1} :: (\ast \rightarrow \ast \rightarrow \ast) \rightarrow (\ast \rightarrow \ast \rightarrow \ast) \rightarrow \ast \]

If we had tuples on the kind level:

\[ \text{Fix}_2 :: (\ast^2 \rightarrow \ast)^2 \rightarrow \ast^2 \]

And if we had numbers as kinds:

\[ \text{Fix}_2 :: ((2 \rightarrow \ast) \rightarrow (2 \rightarrow \ast)) \rightarrow (2 \rightarrow \ast) \]
Kinds (contd.)

\[ \text{Fix}_{2,0} :: (\ast \to \ast \to \ast) \to (\ast \to \ast \to \ast) \to \ast \]
\[ \text{Fix}_{2,1} :: (\ast \to \ast \to \ast) \to (\ast \to \ast \to \ast) \to \ast \]

If we had tuples on the kind level:

\[ \text{Fix}_2 :: (\ast^2 \to \ast)^2 \to \ast^2 \]

And if we had numbers as kinds:

\[ \text{Fix}_2 :: ((2 \to \ast) \to (2 \to \ast)) \to (2 \to \ast) \]

And this can be generalized:

\[ \text{Fix}_n :: ((n \to \ast) \to (n \to \ast)) \to (n \to \ast) \]
One fixed point combinator

\[ \text{Fix}_n :: ((n \to *) \to (n \to *)) \to (n \to *) \]

Can we express \( n \) in Haskell?

Yes!
One fixed point combinator

\[ \text{Fix}_n :: ((n \to *) \to (n \to *)) \to (n \to *) \]

Can we express \( n \) in Haskell?

Yes!
Encoding kind \( n \)

- Choose \( \ast \) rather than \( n \).
Encoding kind $n$

- Choose $\ast$ rather than $n$.
- Ensure that wherever $\ast$ is used instead of $n$, we only instantiate it with one of $n$ different types – the types that make up our family.

$\forall _{ix::n} \rightarrow \ldots$ becomes $\forall _{ix::\ast}Fam_{ix} \rightarrow \ldots$
Encoding kind \( n \)

- Choose \( \ast \) rather than \( n \).
- Ensure that wherever \( \ast \) is used instead of \( n \), we only instantiate it with one of \( n \) different types – the types that make up our family.
- Where necessary, provide additional evidence (in the form of a GADT) that the type is actually one of only \( n \) different possibilities.
Encoding kind n

- Choose \( \ast \) rather than \( n \).
- Ensure that wherever \( \ast \) is used instead of \( n \), we only instantiate it with one of \( n \) different types – the types that make up our family.
- Where necessary, provide additional evidence (in the form of a GADT) that the type is actually one of only \( n \) different possibilities.

\[
\forall \text{ix} :: n. \ldots
\]

becomes

\[
\forall \text{ix} :: \ast. \text{Fam} \text{ ix} \rightarrow \ldots
\]
Example index GADT

\textbf{data} Fam :: \ast \rightarrow \ast \textbf{ where}

\hspace{1em} Expr :: Fam Expr

\hspace{1em} Op :: Fam Op

A value of Fam t encodes a proof that t is either Expr or Op.
Representing a family

```
data ExprF e o =
    ConstF Val
    | IfF e e e
    | BinF e o e

data OpF e o =
    AddF | MulF | InfixF e
    | FlipF o
```
Representing a family

```haskell
data ExprF (r :: * → *) (ix :: *) =
  ConstF Val
  | IfF (r Expr) (r Expr) (r Expr)
  | BinF (r Expr) (r Op) (r Expr)

data OpF (r :: * → *) (ix :: *) =
  AddF | MulF | InfixF (r Expr) | FlipF (r Op)
```
Representing a family

\[
\textbf{data} \ \text{ExprF} \ (r :: \ast \to \ast) \ (ix :: \ast) = \\
\quad \text{ConstF} \ \text{Val} \\
\quad | \quad \text{IfF} \ (r \ \text{Expr}) \ (r \ \text{Expr}) \ (r \ \text{Expr}) \\
\quad | \quad \text{BinF} \ (r \ \text{Expr}) \ (r \ \text{Op}) \ (r \ \text{Expr})
\]

\[
\textbf{data} \ \text{OpF} \ (r :: \ast \to \ast) \ (ix :: \ast) = \\
\quad \text{AddF} \mid \quad \text{MulF} \mid \quad \text{InfixF} \ (r \ \text{Expr}) \mid \quad \text{FlipF} \ (r \ \text{Op})
\]

\[
\textbf{data} \ \text{FamF} \ (r :: \ast \to \ast) \ (ix :: \ast) \ \text{where} \\
\quad \text{ExprF} :: \text{ExprF} \ r \ \text{Expr} \to \ \text{FamF} \ r \ \text{Expr} \\
\quad | \quad \text{OpF} :: \text{OpF} \ r \ \text{Op} \to \ \text{FamF} \ r \ \text{Op}
\]

\[
\text{type} \ \text{Expr'} = \text{Fix} \ \text{FamF} \ \text{Expr} \\
\text{type} \ \text{Op'} = \text{Fix} \ \text{FamF} \ \text{Op}
\]
Representing a family

**type** ExprF =
  K Val

**type** OpF =

**data** FamF (r :: * → *) (ix :: *) where
  ExprF :: ExprF r Expr → FamF r Expr
  | OpF :: OpF r Op → FamF r Op

**type** Expr' = Fix FamF Expr
**type** Op' = Fix FamF Op
Representing a family

\[
\text{type } \text{ExprF} \quad = \\
\quad K \text{ Val} \\
\quad ::= \quad \text{I Expr} \quad :\times: \quad \text{I Expr} \quad :\times: \quad \text{I Expr} \\
\quad ::= \quad \text{I Expr} \quad :\times: \quad \text{I Op} \quad :\times: \quad \text{I Expr} \\
\text{type } \text{OpF} \quad = \\
\quad U \quad ::= U \quad ::= \quad \text{I Expr} \quad ::= \quad \text{I Op} \\
\text{type } \text{FamF} \quad = \\
\quad \text{ExprF} \quad ::= \quad \text{Expr} \\
\quad ::= \quad \text{OpF} \quad ::= \quad \text{Op} \\
\text{type } \text{Expr'} = \text{Fix FamF Expr} \\
\text{type } \text{Op'} = \text{Fix FamF Op}
\]
Combinators for functors

Recurring on a particular index

```
data l (ix' :: *) (r :: * → *) (ix :: *) = l (r ix')
```

Selecting a particular index

```
data (f :▷: ix') (r :: * → *) (ix :: *) where
  Tag :: f r ix' → (f :▷: ix') r ix'
```
class HFunctor fam (f :: (* → *) → * → *) where
  hmap :: ∀r r'.
        (∀ix.fam ix → r ix → r' ix) →
        (∀ix.fam ix → f r ix → f r' ix)
class HFunctor fam (f :: (∗ → ∗) → ∗ → ∗) where

hmap :: ∀r r'.
     (∀ix.fam ix → r ix → r' ix) →
     (∀ix.fam ix → f r ix → f r' ix)

fold :: ∀fam f r. HFunctor fam f ⇒
     (∀ix.fam ix → f r ix → r ix) →
     (∀ix.fam ix → Fix f ix → r ix)
In the paper or the library

Details

- Conversion between original family and representation.
- Generic function code.
In the paper or the library

Details

- Conversion between original family and representation.
- Generic function code.

Applications

- Variants of folds.
- Classic examples: show, equality.
- Type-indexed datatypes: the zipper.
- Generic rewriting.
Try it

On Hackage

multirec – library described in the paper
zipper – generic zippers based on multirec
regular – single-datatype version of the library