Programming with Universes, Generically

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Well-Typed LLP

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An introduction to Agda
Agda

- Functional programming language
- Static types
- Dependent types
- Pure (explicit effects)
- Total (mostly)
Agda

- Functional programming language
- Static types
- Dependent types
- Pure (explicit effects)
- Total (mostly)

- Actively developed at Chalmers University
- Ulf Norell and many others
- Written in Haskell
Superficially looks a bit like Haskell:

```agda
data N : Set where
    zero : N
    suc : N → N

_+_ : N → N → N
zero + n  =  n
suc m + n = suc (m + n)
```
Superficially looks a bit like Haskell:

```agda
data List (A : Set) : Set where
  []    : List A
  _∷_   : A → List A → List A
map : {A B : Set} → (A → B) → List A → List B
map f []       = []
map f (x :: xs) = f x :: map f xs
```
Dependent types

Types can depend on terms:

```haskell
data Vec (A : Set) : ℕ → Set where
  [ ]    : Vec A zero
  _::_   : {n : ℕ} → A → Vec A n → Vec A (suc n)
  _++_   : {A : Set} {m n : ℕ} →
            Vec A m → Vec A n → Vec A (m + n)

[ ]    ++ ys = ys
(x :: xs) ++ ys = x :: (xs ++ ys)
```

Computation during type-checking

When are two types equal?
Dependent types

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      Vec A m → Vec A n → Vec A (m + n)
[]     + ys = ys
(xs :: xs) + ys = x :: (xs ++ ys)
```

- Computation during type-checking
- When are two types equal?
Type equality

Are these equal?

\[
\text{Vec } \mathbb{N} (2 + 2) \\
\text{Vec } \mathbb{N} 4
\]
Type equality

Are these equal?

\[
\text{Vec } \mathbb{N} \ (2 + 2) \\
\text{Vec } \mathbb{N} \ 4
\]

And these?

\[
\text{Vec } \mathbb{N} \ (n + 1) \\
\text{Vec } \mathbb{N} \ (1 + n)
\]

Simple rule: types are reduced according to their definitions as far as possible and then checked for (alpha-)equality.

▶ \(2 + 2\) reduces to \(4\), so the first two are equal.

▶ \(n + 1\) is stuck, because \(\_ + \_\) is defined by induction on the first argument. The second two are not equal.
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Programs and Proofs
Function in Agda are supposed to be \textbf{total}:

- defined on all inputs,
- terminating.
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- defined on all inputs,
- terminating.

So, despite computation on the type level, type checking is decidable.
Enforcing totality

Relatively simple (but conservative) checks:
- Case distinctions have to be exhaustive.
- Recursion only on structurally smaller terms.
- Datatypes must be strictly positive.

Consequence: Agda has uninhabited types! `data ⊥ : Set` where no constructors, thus no way to construct values of type `⊥`.

Whereas in Haskell:
```
loop :: forall a. a
loop x = x
```
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loop x = x
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Agda becomes interesting as a **logic**.

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Curry-Howard isomorphism

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<td>dependent pair</td>
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A type representing equality

```agda
data _≡_ {A : Set} : A → A → Set where
  refl : {x : A} → x ≡ x
```

The value `refl` is a **witness** that two terms of type `A` are actually equal.
A type representing equality

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```

The value `refl` is a **witness** that two terms of type `A` are actually equal.

Where Agda’s built-in (definitional) equality isn’t enough, we can explicitly prove (and use) equality using `≡`. 
Equality is a congruence:

\[
\text{cong} : \{ A \ B : \text{Set} \} \{ x \ y : A \} \rightarrow \\
(f : A \rightarrow B) \rightarrow x \equiv y \rightarrow f \ x \equiv f \ y \\
\text{cong} f \ \text{refl} = \ \text{refl}
\]
Programming is proving

Equality is a congruence:

\[
\text{cong} : \{ A, B : \text{Set} \} \{ x, y : A \} \rightarrow \\
(f : A \rightarrow B) \rightarrow x \equiv y \rightarrow f \cdot x \equiv f \cdot y \\
\text{cong } f \text{ refl } = \text{ refl}
\]

Zero is a right-unit of addition:

\[
\text{example} : (n : \mathbb{N}) \rightarrow (n + \text{zero}) \equiv n \\
\text{example} \text{ zero } = \text{ refl} \\
\text{example} \ (\text{suc } n) = \text{ cong } \text{suc} \ (\text{example } n)
\]
Programming is proving

Equality is a congruence:

\[
\text{cong} : \{ A, B : \text{Set}\} \{ x, y : A \} \rightarrow \\
(f : A \rightarrow B) \rightarrow x \equiv y \rightarrow f(x) \equiv f(y) \\
\text{cong } f \text{ refl} = \text{refl}
\]

Zero is a right-unit of addition:

\[
\text{example} : (n : \mathbb{N}) \rightarrow (n + \text{zero}) \equiv n \\
\text{example zero} = \text{refl} \\
\text{example } (\text{suc } n) = \text{cong } \text{suc} \text{ (example } n)
\]

Proofs are easier to do **interactively**.
Dependently typed programming

- Programming with data that maintains complex invariants, verified by the type checker.
- Stating and proving properties about programs within the program, using the same language.
- Using precise types to guide the programming process.
Datatype-genericity
Reuse vs. type safety

Types make things different.

They sometimes seem to stand in the way of code reuse.
Reuse vs. type safety

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They sometimes seem to stand in the way of code reuse.

A lot of this tension is already addressed by polymorphism, which now corresponds to universal quantification.

But what if we want different (but related) behaviour for different types?
Datatype-generic programs

Datatype-generic programs allow you to inspect the structure of datatypes while defining a function.
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Classic examples:

- structural equality, structural ordering
- serialization, deserialization
- parsing, pretty-printing
- mapping, traversing, transforming, querying
Historical context

- Active research topic since about 15 years.
- A lot of promising approaches, many based on Haskell.
- Related to OO design patterns such as Visitor and Iterator, but also to techniques such as model-driven design.
- Related to meta-programming, but with the goal to be type-safe.
- Historically required significant language extensions or a preprocessor.
- Advances in FP type systems have made it possible to develop datatype-generic programs (nearly) directly in Haskell.
- Dependent types are even more powerful than current Haskell, so DGP in Agda should be easy . . .
Universes
A *universe* is a type of codes together with an interpretation function that computes types from codes:

\[
\text{Code} : \text{Set} \\
\llbracket \_ \rrbracket : \text{Code} \rightarrow \text{Set}
\]

We cannot inspect types directly.

But we can inspect codes!
Universe

A **universe** is a type of codes together with an interpretation function that computes types from codes:

\[
\text{Code} : \text{Set} \\
[\_ ] : \text{Code} \rightarrow \text{Set}
\]

We cannot inspect types directly.

But we can inspect codes!

A **(datatype-)generic function** is a function defined by induction on the codes:

\[
gf : (C : \text{Code}) \rightarrow \ldots [ C ] \ldots
\]
A simple example

We have *already* seen a universe.
We have already seen a universe.

```haskell
data Vec (A : Set) : \mathbb{N} \to \text{Set} where

[ ] : Vec A zero
_-∷-_ : \{n : \mathbb{N}\} \to A \to Vec A n \to Vec A (\text{suc } n)
```

Here, \(\mathbb{N}\) are the codes, and \text{Vec} is an \(A\)-indexed family of interpretation functions.
A simple example

We have **already** seen a universe.

```haskell
data Vec (A : Set) : ℕ → Set where
  [ ]    : Vec A zero
  _∷_   : {n : ℕ} → A → Vec A n → Vec A (suc n)
```

Here, \(\mathbb{N}\) are the codes, and \(\text{Vec}\) is an \(A\)-indexed family of interpretation functions.

Thus \(\_ + \_\) is a generic function on this universe.
A different way to define the vector universe

Again, we use $\mathbb{N}$ as type of codes.

Vec : (A : Set) → $\mathbb{N}$ → Set

Vec A zero = ⊤

Vec A (suc n) = A × Vec A n
Another interpretation for natural numbers

With $\text{Fin}$, we define the family of finite types:

$\text{Fin} : \mathbb{N} \rightarrow \text{Set}$

$\text{Fin \ zero} = \bot$

$\text{Fin \ (suc \ n)} = \top \cup \text{Fin \ n}$
Another interpretation for natural numbers

With \( \text{Fin} \), we define the family of finite types:

\[
\begin{align*}
\text{Fin} &: \mathbb{N} \to \text{Set} \\
\text{Fin zero} &= \bot \\
\text{Fin (suc n)} &= \top \uplus \text{Fin n}
\end{align*}
\]

Safe lookup:

\[
\begin{align*}
\text{lookup} &: \{ A : \text{Set} \} \to (n : \mathbb{N}) \to \text{Fin n} \to \text{Vec A n} \to A \\
\text{lookup zero} &: () \to A \\
\text{lookup (suc n) (inj1 tt)} &: (x : A, xs) \to x \\
\text{lookup (suc n) (inj2 i1)} &: (x : A, xs) \to \text{lookup n i xs}
\end{align*}
\]
Another definition for finite types

Finite types are closed under union and cartesian product:

```haskell
data Code : Set where
c0 : Code
c1 : Code
_⊕_ : Code → Code → Code
_⊗_ : Code → Code → Code
```

\[
\begin{align*}
[\_] & : \text{Code} \rightarrow \text{Set} \\
[\ c0 \ ] &= \bot \\
[\ c1 \ ] &= \top \\
[\ C \oplus D \ ] &= [\ C \ ] \uplus [\ D \ ] \\
[\ C \otimes D \ ] &= [\ C \ ] \times [\ D \ ]
\end{align*}
\]
Generic equality on finite types

\[ _{\equiv\equiv} : (C : \text{Code}) \rightarrow [C] \rightarrow [C] \rightarrow \text{Bool} \]

\[ _{\equiv\equiv} c0 () () \]

\[ _{\equiv\equiv} c1 \quad \text{tt tt } = \quad \text{true} \]

\[ _{\equiv\equiv} (C \oplus D) (\text{inj}_1 x_1) (\text{inj}_1 x_2) = _{\equiv\equiv} C x_1 x_2 \]

\[ _{\equiv\equiv} (C \oplus D) (\text{inj}_2 y_1) (\text{inj}_2 y_2) = _{\equiv\equiv} D y_1 y_2 \]

\[ _{\equiv\equiv} (C \oplus D) \_ \_ = \text{false} \]

\[ _{\equiv\equiv} (C \otimes D) (x_1, y_1) (x_2, y_2) = _{\equiv\equiv} C x_1 x_2 \land _{\equiv\equiv} D y_1 y_2 \]
Relations between universes

size : Code → ℕ
size c0    = 0
size c1    = 1
size (C ⊕ D) = size C + size D
size (C ⊗ D) = size C * size D

We can prove (in Agda) that the two definitions of finite types are related:

fromFin : (n : ℕ) → Fin n → \([\text{natCode } n]\)
toFin   : (C : Code) → \([C]\) → Fin (size C)
toFromId : \{n : ℕ\} (i : Fin n) → i ≡ toFin (natCode n) (fromFin i)
Adding recursion

A universe for polynomial functors:

```
data Code : Set where
  c0  : Code
  c1  : Code
  _⊕_ : Code → Code → Code
  _⊗_ : Code → Code → Code
  rec : Code
```

We interpret codes as **type constructors** now:

```
[ _ ] : Code → Set → Set
[ c0 ] X = ⊥
[ c1 ] X = ⊤
[ C ⊕ D ] X = [ C ] X ⊔ [ D ] X
[ C ⊗ D ] X = [ C ] X ⊓ [ D ] X
[ rec ] X = X
```
Mapping over functors

We traverse the structure, only modifying parameter positions:

\[
\begin{align*}
\text{map} : \{ X \rightarrow Y : \text{Set} \} & \rightarrow (C : \text{Code}) \rightarrow \\
\quad & (X \rightarrow Y) \rightarrow \ll X \rightarrow C \rr X \rightarrow \ll Y \rightarrow C \rr Y \\
\text{map} \; c_0 & \quad f \; () = \; tt \\
\text{map} \; c_1 & \quad f \; tt = \; tt \\
\text{map} \; (C \oplus D) & \quad f \; (\text{inj}_1 \; x) = \; \text{inj}_1 \; (\text{map} \; C \; f \; x) \\
\text{map} \; (C \oplus D) & \quad f \; (\text{inj}_2 \; y) = \; \text{inj}_2 \; (\text{map} \; D \; f \; y) \\
\text{map} \; (C \otimes D) & \quad f \; (x, y) = \; \text{map} \; C \; f \; x, \; \text{map} \; D \; f \; y \\
\text{map rec} & \quad f \; x = \; f \; x
\end{align*}
\]
Taking fixed points

We plug in the data structure itself for the parameter position:

```haskell
data μ (C : Code) : Set where
  ⟨_⟩ : [[ C ]] (μ C) → μ C
```

Example: Binary trees.

```haskell
BinTree = μ (c1 ⊕ (rec ⊗ rec))
leaf = ⟨inj 1 tt⟩
true = BinTree → BinTree → BinTree
true l r = ⟨inj 2 (l, r)⟩
```
Taking fixed points

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data μ (C : Code) : Set where
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```

Example: Binary trees.

```
BinTree : Set
BinTree = μ (c1 ⊕ (rec ⊗ rec))
leaf : BinTree
leaf = ⟨ inj₁ tt ⟩
true : BinTree → BinTree → BinTree
true l r = ⟨ inj₂ (l , r) ⟩
```
Generically traversing a recursive structure

cata : \{ C : \text{Code} \} \{ X : \text{Set} \} \rightarrow ([C] X \rightarrow X) \rightarrow \mu C \rightarrow X

cata \{ C \} \phi \langle x \rangle = \phi (\text{map} C (\text{cata} \phi) x)

\text{height} : \text{BinTree} \rightarrow \mathbb{N}
\text{height} = \text{cata}[\text{const 0}, \lambda (x, y) \rightarrow 1 + \text{max} x y]
What’s next?

Many approaches that have been tried in Haskell over the years are similar to the one we have just seen:

- **regular** library
- **PolyP** adds a parameter slot (so we can model lists, labelled trees, etc.)
- **multirec** library adds an index to the `rec` constructor, so that we can define fixed points of mutually recursive types.
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▶ regular library
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▶ multirec library adds an index to the rec constructor, so that we can define fixed points of mutually recursive types

Agda helps us:
▶ to relate and understand all these approaches,
▶ to generalize even further,
▶ to prove properties of the resulting generic functions.
How many universes do we need?

Without dependent types (Haskell):

- Defining one universe is ok, but mapping between universes is infeasible.
- We have to decide which representation we want to use.
- But the choice is difficult.
- Simple universes represent less types but allow more functions to be defined.

With Agda, we do not have to decide:

- We can define functions generically over a “suitable” universe.
- We can change representations as needed.
- Ideally, we’d model the complete **data** construct as a universe (**levitation**).
Dependent types encourage us to make more distinctions than we are used to make:

- lists,
- vectors,
- sorted lists,
- lists with an even number of elements,
- lists containing only even numbers.

All of these become different types, yet we still want to perform similar operations.
More generic tools

We need more generic tools and special-purpose universes. Examples:

- Many list-like structures can be represented as reflexive transitive closures of suitable binary relations.
- We can relate unconstrained data structures such as lists to constrained data structures such as vectors by a generic process called algebraic ornamentation.
Conclusions

- Datatype-generic programming allows code to be reused more often.
- Generic functions are very abstract, but the types help you to write them.
- The stronger the type system, the more important (but also the easier) generic programming becomes.
- With dependent types, generic programming is just (ordinary) programming.
- Developing dependently typed generic programs is fun.

Thanks for listening – Questions?